# On the exact formula for neutrino oscillation probability by Kimura, Takamura and Yokomakura

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## Abstract

The exact formula for the neutrino oscillation probability in matter with constant density, which was discovered by Kimura, Takamura and Yokomakura, has been applied mostly to the standard case with three flavor neutrino so far. In this paper applications of their formula to more general cases are discussed. It is shown that this formalism can be generalized to various cases where the matter potential have off-diagonal components, and the two non-trivial examples are given: the case with magnetic moments and a magnetic field and the case with non-standard interactions. It is pointed out that their formalism can be applied also to the case in the long baseline limit with matter whose density varies adiabatically as in the case of solar neutrino.

PACS numbers: 14.60.Pq, 14.60.St

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## I. INTRODUCTION

Neutrino oscillations in matter (See, e.g., Ref. [1] for review.) have been discussed by many people in the past because the oscillation probability has non-trivial behaviors in matter and due to the matter effect it may exhibit non-trivial enhancement which could be physically important. Unfortunately, it is not easy to get an analytical formula for the oscillation probability in the three flavor neutrino scheme in matter, and investigation of its behaviors has been a difficult but important problem in the phenomenology of neutrino oscillations. In 2002 Kimura, Takamura and Yokomakura derived a nice compact formula [2, 3] for the neutrino oscillation probability in matter with constant density. Basically what they showed is that the quantity $\tilde{U}^*_{\alpha j}\tilde{U}_{\beta j}$, which is a factor crucial to express the oscillation probability analytically, can be expressed as a linear combination of $U^*_{\alpha j}U_{\beta j}$, where $\tilde{U}_{\alpha j}$ and $U_{\alpha j}$ stand for the matrix element of the MNS matrix in matter and in vacuum, respectively.

However, their formula is only applicable to the standard three flavor case. In this paper we show that their result can be generalized to various cases. We also show that their formalism can be applied also to the case with slowly varying matter density in the limit of the long neutrino path. In Sect. II, we review briefly some aspects of the oscillation probabilities, including a simple derivation for the formula by Kimura, Takamura and Yokomakura which was given in Ref. [4], because these are used in the following sections. Their formalism is generalized to the various cases where the matter potential has off-diagonal components, and we will discuss the case with large magnetic moments and a magnetic field (Sect. III) and the case with non-standard interactions (Sect. IV). In Sect. V we summarize our conclusions.

## II. GENERALITIES ABOUT OSCILLATION PROBABILITIES

### A. The case of constant density

It has been known [5] (See also earlier works [6, 7, 8].) that after eliminating the negative energy states by a Tani-Foldy-Wouthusen-type transformation, the Dirac equation for neutrinos propagating in matter is reduced to the familiar form:

$$i\frac{d\Psi}{dt} = \left[U\mathcal{E}U^{-1} + \mathcal{A}(t)\right]\Psi, \tag{1}$$

where

$$\begin{aligned}\mathcal{E} &\equiv \mathrm{diag}\left(E_1, E_2, E_3\right), \\ \mathcal{A}(t) &\equiv \sqrt{2}G_F\mathrm{diag}\left(N_e(t) - N_n(t)/2, -N_n(t)/2, -N_n(t)/2\right),\end{aligned}$$

$\Psi^T \equiv (\nu_e, \nu_\mu, \nu_\tau)$ is the flavor eigenstate, $U$ is the Maki-Nakagawa-Sakata (MNS) matrix, $E_j \equiv \sqrt{m_j^2 + \vec{p}^{\,2}}$ $(j = 1, 2, 3)$ is the energy eigenvalue of each mass eigenstate, and the matter effect $\mathcal{A}(t)$ at time (or position ) $t$ is characterized by the density $N_e(t)$ of electrons and the one $N_n(t)$ of neutrons, respectively. Throughout this paper we assume for simplicity that the density of matter is either constant or slowly varying so that its derivative is negligible. The $3 \times 3$ matrix on the right hand side of Eq. (1) can be formally diagonalized as:

$$U\mathcal{E}U^{-1} + \mathcal{A}(t) = \tilde{U}(t)\tilde{\mathcal{E}}(t)\tilde{U}^{-1}(t), \tag{2}$$

where

$$\tilde{\mathcal{E}}(t) \equiv \text{diag}\left(\tilde{E}_1(t), \tilde{E}_2(t), \tilde{E}_3(t)\right)$$

is a diagonal matrix with the energy eigenvalues $\tilde{E}_j(t)$ in the presence of the matter effect.

First of all, let us assume that the matter density $\mathcal{A}(t)$ is constant. Then all the $t$ dependence disappears and Eq. (1) can be easily solved, resulting the flavor eigenstate at the distance $L$:

$$\Psi(L) = \tilde{U} \exp\left(-i\tilde{\mathcal{E}} L\right) \tilde{U}^{-1} \Psi(0). \tag{3}$$

Thus the oscillation probability $P(\nu_\alpha \to \nu_\beta)$ is given by

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \left| \left[ \tilde{U} \exp\left(-i\mathcal{E} L\right) \tilde{U}^{-1} \right]_{\beta\alpha} \right|^2 \\ &= \delta_{\alpha\beta} - 4 \sum_{j<k} \text{Re}\left( \tilde{X}_j^{\alpha\beta} \tilde{X}_k^{\alpha\beta *} \right) \sin^2\left( \frac{\Delta\tilde{E}_{jk} L}{2} \right) \\ &\quad +2 \sum_{j<k} \text{Im}\left( \tilde{X}_j^{\alpha\beta} \tilde{X}_k^{\alpha\beta *} \right) \sin\left( \Delta\tilde{E}_{jk} L \right), \end{aligned} \tag{4}$$

where we have defined

$$\begin{aligned} \tilde{X}_j^{\alpha\beta} &\equiv \tilde{U}_{\alpha j} \tilde{U}_{\beta j}^*, \\ \Delta\tilde{E}_{jk} &\equiv \tilde{E}_j - \tilde{E}_k, \end{aligned}$$

and throughout this paper the indices $\alpha, \beta = (e, \mu, \tau)$ and $j, k = (1, 2, 3)$ stand for those of the flavor and mass eigenstates, respectively. Once we know the eigenvalues $\tilde{E}_j$ and the quantity $\tilde{X}_j^{\alpha\beta}$, the oscillation probability can be expressed analytically.[1]

### B. The case of adiabatically varying density

Secondly, let us consider the case where the density of the matter varies adiabatically as in the case of the solar neutrino deficit phenomena. In this case, instead of Eq. (3), we get

$$\Psi(L) = \tilde{U}(L) \exp\left[-i \int_0^L \tilde{\mathcal{E}}(t)\, dt\right] \tilde{U}(0)^{-1} \Psi(0),$$

where $\tilde{U}(0)$ and $\tilde{U}(L)$ stand for the effective mixing matrices at the origin $t = 0$ and at the end point $t = L$. The oscillation probability is given by

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \left| \left[ \tilde{U}(L) \exp\left\{ -i \int_0^L \tilde{\mathcal{E}}(t)\, dt \right\} \tilde{U}(0)^{-1} \right]_{\beta\alpha} \right|^2 \\ &= \sum_{j,k} \tilde{U}(L)_{\beta j} \tilde{U}(L)_{\beta k}^* \tilde{U}(0)_{\alpha j}^* \tilde{U}(0)_{\alpha k} \exp\left[ -i \int_0^L \Delta\tilde{E}(t)_{jk}\, dt \right]. \end{aligned} \tag{5}$$

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[1] In the standard case with three flavors of neutrinos in matter, the energy eigenvalues $\tilde{E}_j$ can be analytically obtained by the root formula for a cubic equation [9]. So the only non-trivial problem in the standard case is to obtain the expression for $\tilde{X}_j^{\alpha\beta}$, and this was done by Kimura, Takamura and Yokomakura [2, 3]. In general cases, however, the analytic expression for $\tilde{E}_j$ is very difficult or impossible to obtain, and we will discuss below only examples in which the analytic expression for $\tilde{E}_j$ is known.

Eq. (5) requires in general the quantity like $\tilde{U}(t)_{\beta j}\tilde{U}^*(t)_{\beta k}$ which has the same flavor index $\beta$ but different mass eigenstate indices $j, k$, and it turns out that the analytical expression for $\tilde{U}(t)_{\beta j}\tilde{U}^*(t)_{\beta k}$ is hard to obtain. However, if the length $L$ of the neutrino path is very large and if $|\int_0^L \Delta\tilde{E}(t)_{jk}\,dt| \gg 1$ is satisfied for $j \neq k$, as in the case of the solar neutrino deficit phenomena, after averaging over rapid oscillations Eq. (5) is reduced to

$$P(\nu_\alpha \to \nu_\beta) \;=\; \sum_j \tilde{X}_j^{\beta\beta}(L)\tilde{X}_j^{\alpha\alpha}(0),$$

where we have defined

$$\tilde{X}_j^{\alpha\alpha}(t) \equiv \left|\tilde{U}(t)_{\alpha j}\right|^2 .$$

In the case of the solar neutrinos deficit process $\nu_e \to \nu_e$ during the daylight, $\tilde{X}_j^{\beta\beta}(L)$ at the end point $t = L$ and $\tilde{X}_j^{\alpha\alpha}(0)$ at the origin $t = 0$ correspond to $X_j^{\beta\beta}$ in vacuum and $[\tilde{X}_j^{\alpha\alpha}]_\odot$ at the center of the Sun, respectively, where

$$\begin{aligned} X_j^{\alpha\beta} &\equiv U_{\alpha j}U^*_{\beta j} \\ \left[\tilde{X}_j^{\alpha\beta}\right]_\odot &\equiv \left[\tilde{U}_{\alpha j}\tilde{U}^*_{\beta j}\right]_\odot \end{aligned}$$

are bilinear products of the elements of the mixing matrices in vacuum and at the center of the Sun, respectively. Thus we obtain

$$P(\nu_e \to \nu_e) \;=\; \sum_j X_j^{ee}\left[\tilde{X}_j^{ee}\right]_\odot .$$

Hence we see that evaluation of the quantity $\tilde{X}_j^{\alpha\alpha}$ in the presence of the matter effect is important not only in the case of constant matter density but also in the case of adiabatically varying density.

### C. Another derivation of the formula by Kimura, Takamura and Yokomakura

In this subsection a systematic derivation of their formula is given because such a derivation will be crucial for the generalizations in the following sections.[2] The arguments are based on the trivial identities. From the unitarity condition of the matrix $\tilde{U}$, we have

$$\delta_{\alpha\beta} = \left[\tilde{U}\tilde{U}^{-1}\right]_{\alpha\beta} = \sum_j \tilde{U}_{\alpha j}\tilde{U}^*_{\beta j} = \sum_j \tilde{X}_j^{\alpha\beta}. \tag{6}$$

Next we take the $(\alpha, \beta)$ component of the both hand sides in Eq. (2):

$$\left[U\mathcal{E}U^{-1} + \mathcal{A}\right]_{\alpha\beta} = \left[\tilde{U}\tilde{\mathcal{E}}\tilde{U}^{-1}\right]_{\alpha\beta} = \sum_j \tilde{U}_{\alpha j}\tilde{E}_j\tilde{U}^*_{\beta j} = \sum_j \tilde{E}_j\tilde{X}_j^{\alpha\beta} \tag{7}$$

[2] The argument here is the same as that in Ref. [4]. Since this derivation does not seem to be widely known, it is reviewed here.

Furthermore, we take the $(\alpha,\beta)$ component of the square of Eq. (2):

$$\left[\left(U\mathcal{E}U^{-1}+\mathcal{A}\right)^2\right]_{\alpha\beta}=\left[\tilde{U}\tilde{\mathcal{E}}^2\tilde{U}^{-1}\right]_{\alpha\beta}=\sum_j\tilde{U}_{\alpha j}\tilde{E}_j^2\tilde{U}^*_{\beta j}=\sum_j\tilde{E}_j^2\tilde{X}_j^{\alpha\beta} \tag{8}$$

Putting Eqs. (6)–(8) together, we have

$$\begin{pmatrix}1&1&1\\ \tilde{E}_1&\tilde{E}_2&\tilde{E}_3\\ \tilde{E}_1^2&\tilde{E}_2^2&\tilde{E}_3^2\end{pmatrix}\begin{pmatrix}\tilde{X}_1^{\alpha\beta}\\ \tilde{X}_2^{\alpha\beta}\\ \tilde{X}_3^{\alpha\beta}\end{pmatrix}=\begin{pmatrix}\delta_{\alpha\beta}\\ \left[U\mathcal{E}U^{-1}+\mathcal{A}\right]_{\alpha\beta}\\ \left[(U\mathcal{E}U^{-1}+\mathcal{A})^2\right]_{\alpha\beta}\end{pmatrix},$$

which can be easily solved by inverting the Vandermonde matrix:

$$\begin{pmatrix}\tilde{X}_1^{\alpha\beta}\\ \tilde{X}_2^{\alpha\beta}\\ \tilde{X}_3^{\alpha\beta}\end{pmatrix}=\begin{pmatrix}\dfrac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(\tilde{E}_2\tilde{E}_3,\ -(\tilde{E}_2+\tilde{E}_3),\ 1)\\ \dfrac{-1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(\tilde{E}_3\tilde{E}_1,\ -(\tilde{E}_3+\tilde{E}_1),\ 1)\\ \dfrac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(\tilde{E}_1\tilde{E}_2,\ -(\tilde{E}_1+\tilde{E}_2),\ 1)\end{pmatrix}\begin{pmatrix}\delta_{\alpha\beta}\\ \left[U\mathcal{E}U^{-1}+\mathcal{A}\right]_{\alpha\beta}\\ \left[(U\mathcal{E}U^{-1}+\mathcal{A})^2\right]_{\alpha\beta}\end{pmatrix}. \tag{9}$$

$[(U\mathcal{E}U^{-1}+\mathcal{A})^j]_{\alpha\beta}$ $(j=1,2)$ on the right hand side are given by the known quantities:

$$\begin{aligned}\left[U\mathcal{E}U^{-1}+\mathcal{A}\right]_{\alpha\beta}&=\sum_j E_jX_j^{\alpha\beta}+A\,\delta_{\alpha e}\delta_{\beta e}\\ \left[\left(U\mathcal{E}U^{-1}+\mathcal{A}\right)^2\right]_{\alpha\beta}&=\sum_j E_j^2X_j^{\alpha\beta}+A\sum_j E_j\left(\delta_{\alpha e}X_j^{e\beta}+\delta_{\beta e}X_j^{\alpha e}\right)+A^2\,\delta_{\alpha e}\delta_{\beta e}.\end{aligned}$$

It can be shown that Eq. (9) coincides with the original results by Kimura, Takamura and Yokomakura [2, 3].

A remark is in order on Eq. (9). Addition of a matrix $c\mathbf{1}$ to Eq. (2) where $c$ is a constant and $\mathbf{1}$ is the identity matrix, or in other words, the shift

$$E_j\to E_j+c\ \ (j=1,2,3), \tag{10}$$

should give the same result for $\tilde{X}_j^{\alpha\beta}$ $(j=1,2,3)$, since Eq. (10) only affects the overall phase of the oscillation amplitude and the phase has to disappear in the probability. It is easy to show that the shift (10) indeed gives the same result as Eq. (9). The proof is given in Appendix A. In practical calculations below, we will always put $c=-E_1$, i.e., we will consider the mass matrix $U(\mathcal{E}-E_1\mathbf{1})U^{-1}+\mathcal{A}$ instead of the original one $U\mathcal{E}U^{-1}+\mathcal{A}$, since all the diagonal elements $(\mathcal{E}-E_1\mathbf{1})_{jj}=\Delta E_{j1}=\Delta m_{j1}^2/2E$ are expressed in terms of the relevant variables $\Delta m_{j1}^2$, and therefore calculations become simpler. To save space, however, we will use the matrix $U\mathcal{E}U^{-1}+\mathcal{A}$ in most of the following discussions.

### D. The case with arbitrary number of neutrinos

It is straightforward to generalize the discussions in sect. II C to the case with arbitrary number of neutrinos where the matter potential is diagonal in the flavor eigenstate. The

scheme with number of sterile neutrinos is one of the example of these cases [4, 10]. The time evolution of such a scheme with $N$ neutrino flavors is described by

$$i\frac{d\Psi_N}{dt} = \left(U_N\mathcal{E}_N U_N^{-1} + \mathcal{A}_N\right)\Psi_N,$$

where $\Psi_N^T \equiv (\nu_{\alpha_1}, \nu_{\alpha_2}, \cdots, \nu_{\alpha_N})$ is the flavor eigenstate,

$$\mathcal{E}_N \;\equiv\; \mathrm{diag}\,(E_1, E_2, \cdots, E_N) \tag{11}$$

is the energy matrix of the mass eigenstate,

$$\mathcal{A}_N \;\equiv\; \mathrm{diag}\,(A_1, A_2, \cdots, A_N)\,,$$

is the potential matrix for the flavor eigenstate, and $U_N$ is the $N\times N$ MNS matrix. As in the previous sect., by taking the $\alpha, \beta$ components, we get

$$\sum_j \tilde{E}_j^m \tilde{X}_j^{\alpha\beta} = \left[\left(U_N\mathcal{E}_N U_N^{-1} + \mathcal{A}_N\right)^m\right]_{\alpha\beta} \quad \text{for } m = 0, \cdots, N-1,$$

which leads to the simultaneous equation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tilde{E}_1 & \tilde{E}_2 & \cdots & \tilde{E}_N \\ \vdots & \vdots & & \vdots \\ \tilde{E}_1^{N-1} & \tilde{E}_2^{N-1} & \cdots & \tilde{E}_N^{N-1} \end{pmatrix} \begin{pmatrix} \tilde{X}_1^{\alpha\beta} \\ \tilde{X}_2^{\alpha\beta} \\ \vdots \\ \tilde{X}_N^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\beta} \\ \left[U_N\mathcal{E}U_N^{-1} + \mathcal{A}_N\right]_{\alpha\beta} \\ \vdots \\ \left[\left(U_N\mathcal{E}U_N^{-1} + \mathcal{A}_N\right)^{N-1}\right]_{\alpha\beta} \end{pmatrix}. \tag{12}$$

Eq. (12) can be solved by inverting the $N\times N$ Vandermonde matrix $V_N$:

$$\begin{pmatrix} \tilde{X}_1^{\alpha\beta} \\ \tilde{X}_2^{\alpha\beta} \\ \vdots \\ \tilde{X}_N^{\alpha\beta} \end{pmatrix} = V_N^{-1} \begin{pmatrix} \delta_{\alpha\beta} \\ \left[U_N\mathcal{E}_N U_N^{-1} + \mathcal{A}_N\right]_{\alpha\beta} \\ \vdots \\ \left[\left(U_N\mathcal{E}_N U_N^{-1} + \mathcal{A}_N\right)^{N-1}\right]_{\alpha\beta} \end{pmatrix}. \tag{13}$$

The determinant of $V_N$ is the Vandermonde determinant $\prod_{j<k}\Delta\tilde{E}_{jk}$, and therefore $V^{-1}$ can be analytically obtained as long as we know the value of $\tilde{E}_j$. The factors $[(U_N\mathcal{E}_N U_N^{-1} + \mathcal{A}_N)^j]_{\alpha\beta}$ on the right hand side of Eq. (13) can be expressed as functions of the energy $E_j$, the quantity $X_j^{\alpha\beta}$ in vacuum and the matter potential $A_\gamma$, since the matrix $(U_N\mathcal{E}_N U_N^{-1}+\mathcal{A}_N)^j$ is a sum of products of the matrices $[(U_N\mathcal{E}_N U_N^{-1})^\ell]_{\gamma\delta} = \sum_k E_j^\ell X_k^{\gamma\delta}$ $(0 \le \ell \le j)$ and $[(\mathcal{A}_N)^m]_{\epsilon\eta} = (A_\epsilon)^m \delta_{\epsilon\eta}$ $(0 \le m \le j)$. From Eq. (13) it is clear that enhancement of the oscillation probability due to the matter effect occurs only when some of $\Delta\tilde{E}_{jk}$ becomes small.

## III. THE CASE WITH LARGE MAGNETIC MOMENTS AND A MAGNETIC FIELD

So far we have assumed that the potential term is diagonal in the flavor basis. We can generalize the present result to the cases where we have off-diagonal potential terms. One of such examples is the case where there are only three active neutrinos with magnetic moments and the magnetic field (See, e.g., Ref. [1] for review.). The hermitian matrix[3]

$$\mathcal{M} \equiv \begin{pmatrix} U\mathcal{E}U^{-1} & \mathcal{B} \\ \mathcal{B}^\dagger & U^*\mathcal{E}(U^*)^{-1} \end{pmatrix} \tag{14}$$

with

$$\mathcal{B} \equiv B\,\mu_{\alpha\beta}$$

is the mass matrix for neutrinos and anti-neutrinos without the matter effect where neutrinos have the magnetic moments $\mu_{\alpha\beta}$ in the magnetic field $B$. Here we assume the magnetic interaction of Majorana type

$$\mu_{\alpha\beta}\bar{\nu}_\alpha\, F_{\lambda\kappa}\sigma^{\lambda\kappa}\,\nu^c_\beta + h.c., \tag{15}$$

and in this case the magnetic moments $\mu_{\alpha\beta}$ are real and anti-symmetric in flavor indices: $\mu_{\alpha\beta} = -\mu_{\beta\alpha}$.

If the magnetic field is constant, then the oscillation probability can be written as

$$\begin{aligned} P(\nu_A \to \nu_B) \ &= \ \delta_{AB} - 4\sum_{J<K} \mathrm{Re}\left(\tilde{X}^{AB}_J \tilde{X}^{AB*}_K\right)\sin^2\left(\frac{\Delta\tilde{E}_{JK}L}{2}\right) \\ &\quad +2\sum_{J<K} \mathrm{Im}\left(\tilde{X}^{AB}_J \tilde{X}^{AB*}_K\right)\sin\left(\Delta\tilde{E}_{JK}L\right), \end{aligned} \tag{16}$$

where $A, B$ run $e, \mu, \tau, \bar{e}, \bar{\mu}, \bar{\tau}$, and $J, K$ run $1, \cdots, 6$, respectively, and $\tilde{X}^{AB}_J \equiv U_{AJ}U^*_{BJ}$. $\tilde{E}_J$ $(J = 1, \cdots, 6)$ are the eigenvalues of the $6\times 6$ matrix $\mathcal{M}$. On the other hand, if the magnetic field varies very slowly and if the length $L$ of the baseline is so long that $|\Delta\tilde{E}_{JK}L| \gg 1$ is satisfied for $J \neq K$, then the oscillation probability is given by

$$P(\nu_A \to \nu_B) \ = \ \sum_{J=1}^{6} \tilde{X}^{BB}_J(L)\tilde{X}^{AA}_J(0). \tag{17}$$

Following the same arguments as before, the quantity $\tilde{X}^{AB}_J$ is given by inverting the $6\times 6$ Vandermonde matrix $V_6$:

$$\begin{pmatrix} \tilde{X}^{AB}_1 \\ \tilde{X}^{AB}_2 \\ \vdots \\ \tilde{X}^{AB}_6 \end{pmatrix} = V_6^{-1} \begin{pmatrix} \delta_{AB} \\ [\mathcal{M}]_{AB} \\ \vdots \\ \left[(\mathcal{M})^5\right]_{AB} \end{pmatrix}. \tag{18}$$

[3] See [5] for derivation of Eq. (14) from the Dirac Eq.

As in the previous sections, $[(\mathcal{M})^J]_{AB}$ $(J = 0, \cdots, 5)$ on the right hand side of Eq. (18) can be expressed in terms of the known quantities $X_K^{AB}$ and $\mathcal{B}_{CD}$, and Eqs. (16) and (18) are useful only when we know the eigenvalues $\tilde{E}_J$.

To demonstrate the usefulness of these formulae, let us consider the case where the magnetic field is large at origin but is zero at the end point and the magnetic field varies adiabatically. For simplicity we assume that $\theta_{13}$ and all the CP phases vanish.[4] In this case the $6 \times 6$ matrix $\mathcal{M}$ in Eq. (14) becomes real, and we obtain the following oscillation probabilities:

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = \sum_{j=1}^{3} (U_{\beta j})^2 [\mathrm{Re}\, \tilde{U}(0)_{\alpha j}]^2 \\ P(\nu_\alpha \to \bar{\nu}_\beta) &= P(\bar{\nu}_\alpha \to \nu_\beta) = \sum_{j=1}^{3} (U_{\beta j})^2 [\mathrm{Im}\, \tilde{U}(0)_{\alpha j}]^2, \end{aligned} \tag{19}$$

where $\tilde{U}(0)$ the $3 \times 3$ unitary matrix which diagonalizes the $3 \times 3$ matrix $U\mathcal{E}U^{-1} + i\mathcal{B}(0)$ at the origin:

$$U\mathcal{E}U^{-1} + i\mathcal{B}(0) = \tilde{U}(0)\tilde{\mathcal{E}}(0)\tilde{U}^{-1}(0).$$

In this example the energy eigenvalues are degenerate, i.e., the $6 \times 6$ energy matrix becomes $\mathrm{diag}(\tilde{\mathcal{E}}, \tilde{\mathcal{E}})$, and the oscillation probability differs from Eq. (17) because the condition $|\Delta\tilde{E}_{JK}L| \gg 1 (J \neq K)$ is not satisfied (e.g., $\Delta\tilde{E}_{JK} = 0$ not only for $J = K = 1$ but also for $J = 1$, $K = 4$). Each probability in Eqs. (19) itself is not expressed in terms of $\tilde{X}_j^{\alpha\alpha}(0)$, but we find that the following relation holds:

$$P(\nu_\alpha \to \nu_\beta) + P(\bar{\nu}_\alpha \to \nu_\beta) = \sum_{j=1}^{3} (U_{\beta j})^2 |\tilde{U}(0)_{\alpha j}|^2 = \sum_{j=1}^{3} X_j^{\beta\beta} \tilde{X}_j^{\alpha\alpha}(0). \tag{20}$$

Eq. (20) is a new result and without the present formalism it would be hard to derive it. The details of derivation of Eq. (19) and explicit forms of $\tilde{X}_j^{\alpha\alpha}(0)$ are given in Appendix B. Eq. (20) may be applicable to the case where high energy astrophysical neutrinos, which are produced in a relatively large magnetic field, are observed on the Earth, on the assumption that the fluxes of neutrinos and anti-neutrinos are almost equal.

## IV. THE CASE WITH NON-STANDARD INTERACTIONS

Another interesting application is the oscillation probability in the presence of new physics in propagation [11, 12]. In this case the mass matrix is given by

$$U\mathcal{E}U^{-1} + \mathcal{A}_{NP} \tag{21}$$

[4] In the presence of the magnetic interaction (15) of Majorana type, the two CP phases, which are absorbed by redefinition of the charged lepton fields in the standard case, cannot be absorbed and therefore become physical. Here, however, we will assume for simplicity that these CP phases vanish.

where

$$\mathcal{A}_{NP} \equiv \sqrt{2}G_F N_e \begin{pmatrix} 1+\epsilon_{ee} & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{e\mu}^* & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{e\tau}^* & \epsilon_{\mu\tau}^* & \epsilon_{\tau\tau} \end{pmatrix}.$$

The dimensionless quantities $\epsilon_{\alpha\beta}$ stand for possible deviation from the standard matter effect. Also in this case the oscillation probability is given by Eqs. (4) and (9), where the standard potential matrix $\mathcal{A}$ has to be replaced by $\mathcal{A}_{NP}$. The extra complication compared to the standard case is calculations of the eigenvalues $\tilde{E}_j$ and the elements $[(U\mathcal{E}U^{-1}+\mathcal{A}_{NP})^m]_{\alpha\beta}$ $(m=1,2)$.

Again to demonstrate the usefulness of the formalism, here we will discuss for simplicity the case in which the eigenvalues are the roots of a quadratic equation. It is known [13] that the constraints on the three parameters $\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}$ from various experimental data are weak and they could be as large as $\mathcal{O}(1)$. In Ref. [14] it was found that large values ($\sim \mathcal{O}(1)$) of the parameters $\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}$ are consistent with all the experimental data including those of the atmospheric neutrino data, provided that one of the eigenvalues of the matrix (21) at high energy limit, i.e., $\mathcal{A}_{NP}$, becomes zero. Simplifying even further, here we will neglect the parameters $\epsilon_{e\mu}$, $\epsilon_{\mu\mu}$, $\epsilon_{\mu\tau}$ which are smaller than $\mathcal{O}(10^{-2})$ and we will consider the potential matrix

$$\mathcal{A}_{NP} = A \begin{pmatrix} 1+\epsilon_{ee} & 0 & \epsilon_{e\tau} \\ 0 & 0 & 0 \\ \epsilon_{e\tau}^* & 0 & \epsilon_{\tau\tau} \end{pmatrix}, \tag{22}$$

where $A \equiv \sqrt{2}G_F N_e$, the three parameters $\epsilon_{ee}$, $\epsilon_{e\tau}$, $\epsilon_{\tau\tau}$ are constrained in such a way that two of the three eigenvalues become zero. We will assume that $N_e$ is constant, and we will take the limit $\Delta m^2_{21} \to 0$. The oscillation probability $P(\nu_\mu \to \nu_e)$ in this case can be analytically expressed and is given by

$$\begin{aligned} P(\nu_\mu \to \nu_e) &= -4\mathrm{Re}\left(\tilde{X}_1^{\mu e}\tilde{X}_2^{\mu e *}\right)\sin^2\left(\frac{\Lambda_- L}{2}\right) - 4\mathrm{Re}\left(\tilde{X}_2^{\mu e}\tilde{X}_3^{\mu e *}\right)\sin^2\left(\frac{\Lambda_+ L}{2}\right) \\ &\quad -4\mathrm{Re}\left(\tilde{X}_1^{\mu e}\tilde{X}_3^{\mu e *}\right)\sin^2\left[\frac{(\Lambda_+ - \Lambda_- L)L}{2}\right] \\ &\quad +\frac{8A(\Delta E_{31})^2}{\Lambda_+\Lambda_-(\Lambda_+ - \Lambda_-)}\ |\epsilon_{e\tau}X_3^{e\mu}X_3^{\mu\tau}|\ \sin(\arg(\epsilon_{e\mu})+\delta) \\ &\quad \times\sin\left(\frac{\Lambda_- L}{2}\right)\sin\left(\frac{\Lambda_+ L}{2}\right)\sin\left[\frac{(\Lambda_+ - \Lambda_-)L}{2}\right]. \end{aligned} \tag{23}$$

Eq. (23) is another new result and it would be difficult to obtain it without using the present formalism. The details of derivation of Eq. (23), explanation of the notations and the explicit forms of all the variables in Eq. (23) are described in Appendix C.

## V. CONCLUSIONS

The essence of the exact formula for the neutrino oscillation probability in constant matter which was discovered by Kimura, Takamura and Yokomakura lies in the fact that the combination $\tilde{X}_j^{\alpha\beta} \equiv \tilde{U}^{\alpha j}\tilde{U}^{\beta j *}$ of the mixing matrix elements in matter can be expressed

as polynomials in the same quantity $X_j^{\alpha\beta} \equiv U^{\alpha j} U^{\beta j*}$ in vacuum. In this paper we have discussed applications of their formalism to more general cases. We have pointed out that their formalism can be useful for the cases in matter not only with constant density but also with density which varies adiabatically as in the case of the solar neutrino problem, after taking the limit of the long neutrino path. We have shown that their formalism can be generalized to the cases where the matter potential has off-diagonal components. As concrete non-trivial examples, we discussed the case with magnetic moments and a magnetic field, and the case with non-standard interactions. The application of the present formalism to the case with unitarity violation has been discussed elsewhere [15]. The formalism by Kimura, Takamura and Yokomakura is quite general and can be applicable to many problems in neutrino oscillation phenomenology.

## APPENDIX A: PROOF THAT EQ. (10) GIVES THE SAME (9)

In this appendix we show that Eq. (10) gives the same result for $\tilde{X}_j^{\alpha\beta}$ $(j=1,2,3)$. The value of $\tilde{X}_j^{\alpha\beta}$ $(j=1,2,3)$ for

$$\tilde{U}\left(\tilde{\mathcal{E}} + c\mathbf{1}\right)\tilde{U}^{-1} = U\mathcal{E}U^{-1} + \mathcal{A} + c\mathbf{1}$$

becomes at most quadratic[5] in $c$, and all one has to do is to show that the coefficients of the terms linear and quadratic in $c$ vanish. Let us introduce the notation

$$\begin{aligned}
\left(\begin{array}{ccc} 1 & 1 & 1 \\ \tilde{E}_1 + c & \tilde{E}_2 + c & \tilde{E}_3 + c \\ (\tilde{E}_1 + c)^2 & (\tilde{E}_2 + c)^2 & (\tilde{E}_3 + c)^2 \end{array}\right)^{-1} &\equiv (V^{-1})^{(0)} + c(V^{-1})^{(1)} + c^2(V^{-1})^{(2)} \\
\left(\begin{array}{r} \delta_{\alpha\beta} \\ [U\mathcal{E}U^{-1} + \mathcal{A} + c\mathbf{1}]_{\alpha\beta} \\ \left[(U\mathcal{E}U^{-1} + \mathcal{A} + c\mathbf{1})^2\right]_{\alpha\beta} \end{array}\right) &\equiv \vec{B}^{(0)} + c\vec{B}^{(1)} + c^2\vec{B}^{(2)},
\end{aligned}$$

where $V^{(k)}$ is the coefficient of the inverted Vandermonde matrix which is $k$-th order in $c$, and $B_j^{(k)}$ is the coefficient of the vector $(U\mathcal{E}U^{-1} + \mathcal{A} + c\mathbf{1})^j$ which is $k$-th order in $c$. Then the terms linear in $c$ are given by

$$\begin{aligned}
&(V^{-1})^{(1)}\vec{B}^{(0)} + (V^{-1})^{(0)}\vec{B}^{(1)} \\
&= \left(\begin{array}{l} \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(\tilde{E}_2 + \tilde{E}_3, \quad -2, \ 0) \\ \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(-(\tilde{E}_3 + \tilde{E}_1), \ +2, \ 0) \\ \frac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(\tilde{E}_1 + \tilde{E}_2, \quad -2, \ 0) \end{array}\right)\left(\begin{array}{r} \delta_{\alpha\beta} \\ [U\mathcal{E}U^{-1} + \mathcal{A}]_{\alpha\beta} \\ \left[(U\mathcal{E}U^{-1} + \mathcal{A})^2\right]_{\alpha\beta} \end{array}\right)
\end{aligned}$$

---

[5] Notice that all the factors $\Delta\tilde{E}_{jk}$ are invariant under the shift (10), and the only change by this shift comes either from the terms $\tilde{E}_j\tilde{E}_k$ or from $\tilde{E}_j + \tilde{E}_k$ in the inverse of the Vandermonde matrix (cf. Eq. (9)). Hence the difference by Eq. (10) is at most quadratic in $c$.

$$
+\begin{pmatrix} \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(+\tilde{E}_2\tilde{E}_3,\ -(\tilde{E}_2+\tilde{E}_3),\ +1) \\ \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(-\tilde{E}_3\tilde{E}_1,\ +(\tilde{E}_3+\tilde{E}_1),\ -1) \\ \frac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(+\tilde{E}_1\tilde{E}_2,\ -(\tilde{E}_1+\tilde{E}_2),\ +1) \end{pmatrix} \begin{pmatrix} 0 \\ \delta_{\alpha\beta} \\ 2\left[U\mathcal{E}U^{-1}+\mathcal{A}\right]_{\alpha\beta} \end{pmatrix} = 0,
$$

and the terms quadratic in $c$ are given by

$$
\begin{aligned}
&(V^{-1})^{(2)}\vec{B}^{(0)} + (V^{-1})^{(1)}\vec{B}^{(1)} + (V^{-1})^{(0)}\vec{B}^{(2)} \\
&= \begin{pmatrix} \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(+1,\ 0,\ 0) \\ \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(-1,\ 0,\ 0) \\ \frac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(+1,\ 0,\ 0) \end{pmatrix} \begin{pmatrix} \delta_{\alpha\beta} \\ \left[U\mathcal{E}U^{-1}+\mathcal{A}\right]_{\alpha\beta} \\ \left[(U\mathcal{E}U^{-1}+\mathcal{A})^2\right]_{\alpha\beta} \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(\tilde{E}_2+\tilde{E}_3,\ \ -2,\ 0) \\ \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(-(\tilde{E}_3+\tilde{E}_1),\ +2,\ 0) \\ \frac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(\tilde{E}_1+\tilde{E}_2,\ \ -2,\ 0) \end{pmatrix} \begin{pmatrix} 0 \\ \delta_{\alpha\beta} \\ 2\left[U\mathcal{E}U^{-1}+\mathcal{A}\right]_{\alpha\beta} \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(+\tilde{E}_2\tilde{E}_3,\ -(\tilde{E}_2+\tilde{E}_3),\ +1) \\ \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(-\tilde{E}_3\tilde{E}_1,\ +(\tilde{E}_3+\tilde{E}_1),\ -1) \\ \frac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(+\tilde{E}_1\tilde{E}_2,\ -(\tilde{E}_1+\tilde{E}_2),\ +1) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \delta_{\alpha\beta} \end{pmatrix} = 0.
\end{aligned}
$$

Thus $\tilde{X}_j^{\alpha\beta}$ $(j=1,2,3)$ is independent of $c$, as is claimed.

## APPENDIX B: DERIVATION OF EQ. (19)

The matrix (14) can be rewritten as

$$
\mathcal{M} = \frac{1}{2}\begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ i\mathbf{1} & \mathbf{1} \end{pmatrix}\begin{pmatrix} U\mathcal{E}U^{-1}+i\mathcal{B} & 0 \\ 0 & U\mathcal{E}U^{-1}-i\mathcal{B} \end{pmatrix}\begin{pmatrix} \mathbf{1} & -i\mathbf{1} \\ -i\mathbf{1} & \mathbf{1} \end{pmatrix},
$$

so the problem of diagonalizing the $6\times6$ matrix (14) is reduced to diagonalizing the $3\times3$ matrices $U\mathcal{E}U^{-1}\pm i\mathcal{B}$. Since we are assuming that $\theta_{13}$ and all the CP phases vanish, all the matrix elements $U_{\alpha j}$ and $\mathcal{B}_{\alpha\beta}=-\mathcal{B}_{\beta\alpha}$ are real, $U\mathcal{E}U^{-1}\pm i\mathcal{B}$ can be diagonalized by a unitary matrix and its complex conjugate:

$$
\begin{aligned}
U\mathcal{E}U^{-1}+i\mathcal{B} &= \tilde{U}\tilde{\mathcal{E}}\tilde{U}^{-1} \\
U\mathcal{E}U^{-1}-i\mathcal{B} &= \tilde{U}^*\tilde{\mathcal{E}}(\tilde{U}^*)^{-1}.
\end{aligned}
$$

Therefore, we can diagonalize $\mathcal{M}$ by a $6\times6$ unitary matrix $\tilde{\mathcal{U}}$ as

$$
\mathcal{M} = \tilde{\mathcal{U}}\begin{pmatrix} \tilde{\mathcal{E}} & 0 \\ 0 & \tilde{\mathcal{E}} \end{pmatrix}\tilde{\mathcal{U}}^{-1},
$$

where

$$\tilde{\mathcal{U}} = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} \mathbf{1} & -i\mathbf{1} \\ -i\mathbf{1} & \mathbf{1} \end{array} \right) \left( \begin{array}{cc} \tilde{U} & 0 \\ 0 & \tilde{U}^* \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} \tilde{U} & -i\tilde{U}^* \\ -i\tilde{U} & \tilde{U}^* \end{array} \right).$$

We note in passing that the reason why diagonalization of the $6 \times 6$ matrix is reduced to that of the $3 \times 3$ matrix is because the two matrices $U\mathcal{E}U^{-1}$ and $\mathcal{B}$ are real.

On the other hand, without a magnetic field the $6 \times 6$ unitary matrix $\mathcal{U}$ is given by

$$\mathcal{U} = \left( \begin{array}{cc} U & 0 \\ 0 & U^* \end{array} \right) = \left( \begin{array}{cc} U & 0 \\ 0 & U \end{array} \right),$$

where the CP phase $\delta$ has dropped out because $\theta_{13} = 0$. From these we can integrate the equation of motion and we get the fields at the end point:

$$\begin{aligned} \left( \begin{array}{c} \Psi(L) \\ \Psi^c(L) \end{array} \right) &= \tilde{\mathcal{U}}(L) \left( \begin{array}{cc} e^{-i\Phi} & 0 \\ 0 & e^{-i\Phi} \end{array} \right) \tilde{\mathcal{U}}(0)^{-1} \left( \begin{array}{c} \Psi(0) \\ \Psi^c(0) \end{array} \right) \\ &= \frac{1}{2} \left( \begin{array}{cc} Ue^{-i\Phi}\tilde{U}^{-1} + U^* e^{-i\Phi}(\tilde{U}^*)^{-1} & -i(Ue^{-i\Phi}\tilde{U}^{-1} - U^* e^{-i\Phi}(\tilde{U}^*)^{-1}) \\ i(Ue^{-i\Phi}\tilde{U}^{-1} - U^* e^{-i\Phi}(\tilde{U}^*)^{-1}) & Ue^{-i\Phi}\tilde{U}^{-1} + U^* e^{-i\Phi}(\tilde{U}^*)^{-1} \end{array} \right) \left( \begin{array}{c} \Psi(0) \\ \Psi^c(0) \end{array} \right) \end{aligned}$$

where

$$\Phi \equiv \int_0^L \tilde{\mathcal{E}}(t)\, dt,$$

and we have assumed that a large magnetic field exists at the origin whereas there is no magnetic field at the end point. Thus the oscillation probabilities for the adiabatic transition are give by:

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) &= \lim_{L\to\infty} \left| \frac{1}{2} \left[ Ue^{-i\Phi}\tilde{U}^{-1} + U^* e^{-i\Phi}(\tilde{U}^*)^{-1} \right]_{\alpha\beta} \right|^2 \\ &= \sum_{j=1}^{3} |U_{\beta j}|^2 \left[ \mathrm{Re}(\tilde{U}_{\alpha j}) \right]^2, \\ P(\bar{\nu}_\alpha \to \nu_\beta) = P(\nu_\alpha \to \bar{\nu}_\beta) &= \lim_{L\to\infty} \left| \frac{1}{2} \left[ Ue^{-i\Phi}\tilde{U}^{-1} - U^* e^{-i\Phi}(\tilde{U}^*)^{-1} \right]_{\alpha\beta} \right|^2 \\ &= \sum_{j=1}^{3} |U_{\beta j}|^2 \left[ \mathrm{Im}(\tilde{U}_{\alpha j}) \right]^2. \end{aligned}$$

Hence we obtain the following relation:

$$P(\nu_\alpha \to \nu_\beta) + P(\bar{\nu}_\alpha \to \nu_\beta) = P(\nu_\alpha \to \nu_\beta) + P(\nu_\alpha \to \bar{\nu}_\beta) = \sum_{j=1}^{3} |U_{\beta j}|^2 |\tilde{U}_{\alpha j}|^2.$$

To get $|\tilde{U}_{\alpha j}|^2$, we need the explicit expression for the eigenvalues and the quantity $\tilde{X}_j^{\alpha\alpha}$ in the presence of a magnetic field. In the following we will subtract $E_1\mathbf{1}$ from the energy matrix $\mathcal{E}$ because it will only change the phase of the oscillation amplitude. For simplicity we will put $\theta_{13} = 0$, $\theta_{23} = \pi/4$, and we will consider the limit $\Delta m^2_{21} \to 0$. Defining $\Delta E_{jk} \equiv \Delta m^2_{jk}/2E$ and

$$\mathcal{B}_{\alpha\beta} = B\mu_{\alpha\beta} \equiv \left( \begin{array}{ccc} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{array} \right),$$

we have the eigenvalue equation

$$
\begin{aligned}
0 &= |\lambda \mathbf{1} - U(\mathcal{E} - E_1 \mathbf{1})U^{-1} - i\mathcal{B}| \\
&= \lambda^3 - \Delta E_{31}\lambda^2 - (p^2+q^2+r^2)\lambda + \frac{\Delta E_{31}}{2}(p-q)^2. \qquad \text{(B1)}
\end{aligned}
$$

The three roots of the cubic equation (B1) are given by

$$
\lambda_1 = 2R\cos\varphi + \frac{\Delta E_{31}}{3},\ \lambda_2 = 2R\cos(\varphi + \frac{2}{3}\pi) + \frac{\Delta E_{31}}{3},\ \lambda_3 = 2R\cos(\varphi - \frac{2}{3}\pi) + \frac{\Delta E_{31}}{3},
$$

where

$$
\begin{aligned}
R &\equiv [(\Delta E_{31}/3)^2 + (p^2+q^2+r^2)/3]^{3/2}, \\
\varphi &\equiv (1/3)\cos^{-1}\left[\{(\Delta E_{31}/3)^3 + \Delta E_{31}(p^2+q^2+r^2)/6 - \Delta E_{31}(p-q)^2/4\}/R\right].
\end{aligned}
$$

The quantity $\tilde{X}_j^{\alpha\alpha}$ in the presence of a magnetic field is given by

$$
\begin{pmatrix} \tilde{X}_1^{\alpha\alpha} \\ \tilde{X}_2^{\alpha\alpha} \\ \tilde{X}_3^{\alpha\alpha} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\Delta\lambda_{21}\Delta\lambda_{31}}(\lambda_2\lambda_3,\ -(\lambda_2+\lambda_3),\ 1) \\ \dfrac{-1}{\Delta\lambda_{21}\Delta\lambda_{32}}(\lambda_3\lambda_1,\ -(\lambda_3+\lambda_1),\ 1) \\ \dfrac{1}{\Delta\lambda_{31}\Delta\lambda_{32}}(\lambda_1\lambda_2,\ -(\lambda_1+\lambda_2),\ 1) \end{pmatrix} \begin{pmatrix} 1 \\ Y_2^{\alpha\alpha} \\ Y_3^{\alpha\alpha} \end{pmatrix}, \qquad \text{(B2)}
$$

where

$$
\begin{aligned}
Y_2^{\alpha\alpha} &= \Big[U(\mathcal{E} - E_1\mathbf{1})U^{-1} + i\mathcal{B}\Big]_{\alpha\alpha} = \Delta E_{31} X_3^{\alpha\alpha} \\
&= \begin{cases} 0 & (\alpha = e) \\ \Delta E_{31}/2 & (\alpha = \mu, \tau) \end{cases} \qquad \text{(B3)} \\
Y_3^{\alpha\alpha} &= \Big[\Big\{U(\mathcal{E} - E_1\mathbf{1})U^{-1} + i\mathcal{B}\Big\}^2\Big]_{\alpha\alpha} = (\Delta E_{31})^2 X_3^{\alpha\alpha} - (\mathcal{B}^2)_{\alpha\alpha} \\
&= \begin{cases} q^2 + r^2 & (\alpha = e) \\ r^2 + p^2 + (\Delta E_{31})^2/2 & (\alpha = \mu) \\ p^2 + q^2 + (\Delta E_{31})^2/2 & (\alpha = \tau) \end{cases}. \qquad \text{(B4)}
\end{aligned}
$$

In evaluating $Y_j^{\alpha\alpha}$, we have used the facts $\theta_{13} = 0$, $\theta_{23} = \pi/4$, $\Delta E_{21} = 0$, $\mathcal{B}_{\alpha\beta} = -\mathcal{B}_{\beta\alpha}$, and that $U(\mathcal{E} - E_1\mathbf{1})U^{-1}$ is a symmetric matrix. Using all these results, it is straightforward to obtain the explicit form for $P(\nu_\alpha \to \nu_\beta) + P(\bar{\nu}_\alpha \to \nu_\beta)$ by plugging the results of Eqs. (B2), (B3), (B4) into the following (although calculations are tedious):

$$
\begin{aligned}
P(\nu_\alpha \to \nu_e) + P(\bar{\nu}_\alpha \to \nu_e) &= c_{12}^2 \tilde{X}_1^{\alpha\alpha} + s_{12}^2 \tilde{X}_2^{\alpha\alpha} \\
P(\nu_\alpha \to \nu_\beta) + P(\bar{\nu}_\alpha \to \nu_\beta) &= \frac{c_{12}^2}{2}\tilde{X}_1^{\alpha\alpha} + \frac{s_{12}^2}{2}\tilde{X}_2^{\alpha\alpha} + \frac{1}{2}\tilde{X}_3^{\alpha\alpha} \quad (\beta = \mu, \tau),
\end{aligned}
$$

where $s_{12} \equiv \sin\theta_{12}$, $c_{12} \equiv \cos\theta_{12}$.

## APPENDIX C: DERIVATION OF EQ. (23)

The oscillation probability (23) is obtained in two steps. First we will obtain the eigenvalues of the matrix (21) with Eq. (22) and then we will plug the expressions for the eigenvalues into Eq. (9) with $\mathcal{A}$ replaced by $\mathcal{A}_{NP}$ given in Eq. (22).

Let us introduce notations for $3 \times 3$ hermitian matrices:

$$\begin{aligned}
\lambda_2 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 \equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
\lambda_0 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_9 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\end{aligned}$$

where $\lambda_2$, $\lambda_5$ and $\lambda_7$ are the standard Gell-Mann matrices whereas $\lambda_0$ and $\lambda_9$ are the notations which are defined only in this paper. Simple calculations show that the matrix $\mathcal{A}_{NP}$ in Eq. (22) can be rewritten as

$$\mathcal{A}_{NP} = A\, e^{i\gamma\lambda_9} e^{-i\beta\lambda_5} \left[ \lambda_0 \frac{1+\epsilon_{ee}+\epsilon_{\tau\tau}}{2} + \lambda_9 \sqrt{\left(\frac{1+\epsilon_{ee}-\epsilon_{\tau\tau}}{2}\right)^2 + |\epsilon_{\mu\tau}|^2} \right] e^{i\beta\lambda_5} e^{-i\gamma\lambda_9}, \quad \text{(C1)}$$

where

$$\begin{aligned}
\beta &\equiv \frac{1}{2}\tan^{-1}\frac{2|\epsilon_{e\tau}|^2}{1+\epsilon_{ee}-\epsilon_{\tau\tau}}, \\
\gamma &\equiv \frac{1}{2}\arg\left(\epsilon_{e\mu}\right).
\end{aligned}$$

From Eq. (C1) we see that the two potentially non-zero eigenvalues $\lambda_{e'}$ and $\lambda_{\tau'}$ of the matrix (22) are given by

$$\begin{pmatrix} \lambda_{e'} \\ \lambda_{\tau'} \end{pmatrix} = A \left[ \frac{1+\epsilon_{ee}+\epsilon_{\tau\tau}}{2} \pm \sqrt{\left(\frac{1+\epsilon_{ee}-\epsilon_{\tau\tau}}{2}\right)^2 + |\epsilon_{\mu\tau}|^2} \right].$$

In order for this scheme to be consistent with the atmospheric neutrino data particularly at high energy, which are perfectly described by vacuum oscillations, $\lambda_{\tau'}$ has to vanish [14]. In this case, we have

$$\begin{aligned}
\tan\beta &= \frac{|\epsilon_{e\tau}|}{1+\epsilon_{ee}}, \\
\epsilon_{\tau\tau} &= \frac{|\epsilon_{e\tau}|^2}{1+\epsilon_{ee}}, \\
\lambda_{e'} &= A(1+\epsilon_{ee}) \left[ 1 + \frac{|\epsilon_{e\tau}|^2}{(1+\epsilon_{ee})^2} \right] = \frac{A(1+\epsilon_{ee})}{\cos^2\beta}.
\end{aligned}$$

Thus we have

$$\mathcal{A}_{NP} = A\, e^{i\gamma\lambda_9} e^{-i\beta\lambda_5} \mathrm{diag}\left(\lambda_{e'}, 0, 0\right) e^{i\beta\lambda_5} e^{-i\gamma\lambda_9}. \quad \text{(C2)}$$

If we did not have $\beta$ and $\gamma$, Eq. (C2) would be the same as the standard three flavor scheme in matter, which was analytically worked out in Ref. [16] in the limit of $\Delta m^2_{21} \to 0$. It turns out that, by redefining the parametrization of the MNS matrix Eq. (C2) can be also treated analytically in the limit of $\Delta m^2_{21} \to 0$ as was done in Ref. [16]. The mass matrix can be written as

$$U\mathcal{E}U^{-1} + \mathcal{A}_{NP} = e^{i\gamma\lambda_9}e^{-i\beta\lambda_5}\left[e^{i\beta\lambda_5}e^{-i\gamma\lambda_9}U\mathcal{E}U^{-1}e^{i\gamma\lambda_9}e^{-i\beta\lambda_5} + \text{diag}\left(\lambda_{e'}, 0, 0\right)\right]e^{i\beta\lambda_5}e^{-i\gamma\lambda_9}.$$

Here we introduce the following two unitary matrices:

$$\begin{aligned} U' &\equiv e^{i\beta\lambda_5}e^{-i\gamma\lambda_9}\,U \\ &\equiv \text{diag}(1, 1, e^{i\arg U'_{\tau3}})\,U''\,\text{diag}(e^{i\arg U'_{e1}}, e^{i\arg U'_{e2}}, 1), \end{aligned}$$

where $U$ is the $3 \times 3$ MNS matrix in the standard parametrization [17] and $U''$ was defined in the second line in such a way that the elements $U''_{e1}$, $U''_{e2}$, $U''_{\tau3}$ be real to be consistent with the standard parametrization in Ref. [17] [6]. Then we have

$$\begin{aligned} U\mathcal{E}U^{-1} + \mathcal{A}_{NP} &= e^{i\gamma\lambda_9}e^{-i\beta\lambda_5}\text{diag}(1, 1, e^{i\arg U'_{\tau3}})\left[U''\mathcal{E}U''^{-1} + \text{diag}\left(\lambda_{e'}, 0, 0\right)\right] \\ &\quad\times\text{diag}(1, 1, e^{-i\arg U'_{\tau3}})\,e^{i\beta\lambda_5}e^{-i\gamma\lambda_9}. \end{aligned} \tag{C3}$$

Before proceeding further, let us obtain the expression for the three mixing angles $\theta''_{jk}$ and the Dirac phase $\delta''$ in $U''$. Since

$$U' = \begin{pmatrix} c_\beta e^{-i\gamma}U_{e1} + s_\beta e^{i\gamma}U_{\tau1} & c_\beta e^{-i\gamma}U_{e2} + s_\beta e^{i\gamma}U_{\tau2} & c_\beta e^{-i\gamma}U_{e3} + s_\beta e^{i\gamma}U_{\tau3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ c_\beta e^{-i\gamma}U_{\tau1} - s_\beta e^{i\gamma}U_{e1} & c_\beta e^{-i\gamma}U_{\tau2} - s_\beta e^{i\gamma}U_{e2} & c_\beta e^{-i\gamma}U_{\tau3} - s_\beta e^{i\gamma}U_{e3} \end{pmatrix},$$

where $c_\beta \equiv \cos\beta$, $s_\beta \equiv \sin\beta$, we get

$$\begin{aligned} \theta''_{13} &= \sin^{-1}|U''_{e3}| = \sin^{-1}|c_\beta e^{-i\gamma}U_{e3} + s_\beta e^{i\gamma}U_{\tau3}| \\ \theta''_{12} &= \tan^{-1}(U''_{e2}/U''_{e1}) = \tan^{-1}\left(|c_\beta e^{-i\gamma}U_{e2} + s_\beta e^{i\gamma}U_{\tau2}|/|c_\beta e^{-i\gamma}U_{e1} + s_\beta e^{i\gamma}U_{\tau1}|\right) \\ \theta''_{23} &= \tan^{-1}(U''_{\mu3}/U''_{\tau3}) = \tan^{-1}\left(U_{\mu3}/|c_\beta e^{-i\gamma}U_{\tau3} - s_\beta e^{i\gamma}U_{e3}|\right) \\ \delta'' &= -\arg U''_{e3} = -\arg\left(c_\beta e^{-i\gamma}U_{e3} + s_\beta e^{i\gamma}U_{\tau3}\right). \end{aligned}$$

As was shown in Ref. [16], in the limit $\Delta m^2_{21} \to 0$, the matrix on the right hand side of Eq. (C3) can be diagonalized as follows:

$$\begin{aligned} & U''\mathcal{E}U''^{-1} + \text{diag}\left(\lambda_{e'}, 0, 0\right) - E_1\mathbf{1} \\ &= e^{i\theta''_{23}\lambda_7}\Gamma_{\delta''}e^{i\theta''_{13}\lambda_5}\Gamma^{-1}_{\delta''}e^{i\theta''_{12}\lambda_2}\text{diag}\left(0, 0, \Delta E_{31}\right)e^{-i\theta''_{12}\lambda_2}\Gamma_{\delta''}e^{-i\theta''_{13}\lambda_5}\Gamma^{-1}_{\delta''}e^{-i\theta''_{23}\lambda_7} + \text{diag}\left(\lambda_{e'}, 0, 0\right) \\ &= e^{i\theta''_{23}\lambda_7}\Gamma_{\delta''}\left[e^{i\theta''_{13}\lambda_5}\text{diag}\left(0, 0, \Delta E_{31}\right) + \text{diag}\left(\lambda_{e'}, 0, 0\right)\right]\Gamma^{-1}_{\delta''}e^{-i\theta''_{23}\lambda_7} \\ &= e^{i\theta''_{23}\lambda_7}\Gamma_{\delta''}e^{i\tilde{\theta}''_{13}\lambda_5}\text{diag}\left(\Lambda_-, 0, \Lambda_+\right)e^{-i\tilde{\theta}''_{13}\lambda_5}\Gamma^{-1}_{\delta''}e^{i\theta''_{12}\lambda_2}, \end{aligned}$$

where $\Gamma_{\delta''} \equiv \text{diag}(1, 1, e^{-i\delta''})$, $\Delta E_{31} \equiv \Delta m^2_{31}/2E$, we have used the standard parametrization [17] $U'' \equiv e^{i\theta''_{23}\lambda_7}\Gamma_{\delta''}e^{i\theta''_{13}\lambda_5}\Gamma^{-1}_{\delta''}e^{i\theta''_{12}\lambda_2}$, and the eigenvalues $\Lambda_\pm$ are defined by

$$\Lambda_\pm = \frac{1}{2}\left(\Delta E_{31} + \lambda_{e'}\right) \pm \frac{1}{2}\sqrt{\left(\Delta E_{31}\cos 2\theta''_{13} - \lambda_{e'}\right)^2 + (\Delta E_{31}\sin 2\theta''_{13})^2}.$$

---

[6] The element $U''_{\tau2}$ has to be also real, but it is already satisfied because $U''_{\tau2} = U_{\tau2}$.

Having obtained the eigenvalues, by plugging these into Eq. (9) with $\mathcal{A} \to \mathcal{A}_{NP}$, $\tilde{E}_1 \to \Lambda_-$, $\tilde{E}_2 \to 0$, $\tilde{E}_3 \to \Lambda_+$, we obtain $\tilde{X}^{\mu e}$:

$$\begin{pmatrix} \tilde{X}_1^{\mu e} \\ \tilde{X}_2^{\mu e} \\ \tilde{X}_3^{\mu e} \end{pmatrix} = \begin{pmatrix} \dfrac{-1}{\Lambda_-(\Lambda_+ - \Lambda_-)}(0, & -\Lambda_+, & 1) \\ \dfrac{1}{\Lambda_+\Lambda_-}(-\Lambda_+\Lambda_-, & -(\Lambda_+ + \Lambda_-), & 1) \\ \dfrac{1}{\Lambda_+(\Lambda_+ - \Lambda_-)}(0, & -\Lambda_-, & 1) \end{pmatrix} \begin{pmatrix} 0 \\ Y_2^{\mu e} \\ Y_3^{\mu e} \end{pmatrix} = \begin{pmatrix} \dfrac{-Y_3^{\mu e} + \Lambda_+ Y_2^{\mu e}}{\Lambda_-(\Lambda_+ - \Lambda_-)} \\ \dfrac{Y_3^{\mu e} - (\Lambda_+ + \Lambda_-) Y_2^{\mu e}}{\Lambda_+\Lambda_-} \\ \dfrac{Y_3^{\mu e} - \Lambda_- Y_2^{\mu e}}{\Lambda_+(\Lambda_+ - \Lambda_-)} \end{pmatrix},$$

where $Y_j^{\mu e}$ are defined by

$$Y_j^{\mu e} \equiv \left[ \left( U \mathcal{E} U^{-1} + \mathcal{A}_{NP} \right)^{j-1} \right]_{\mu e},$$

and are given by

$$\begin{aligned} Y_2^{\mu e} &= \Delta E_{31}\, X_3^{\mu e} \\ Y_3^{\mu e} &= [(\Delta E_{31})^2 + A(1+\epsilon_{ee})\Delta E_{31}] X_3^{\mu e} + A \Delta E_{31} \epsilon_{e\tau}^*\, X_3^{\mu\tau}. \end{aligned}$$

Furthermore, by introducing the notations

$$\begin{aligned} \xi &\equiv [(\Delta E_{31})^2 + A(1+\epsilon_{ee})\Delta E_{31}] U_{\mu 3} |U_{e3}| \\ \eta &\equiv A \Delta E_{31} |\epsilon_{e\tau}| U_{\mu 3} U_{\tau 3} \\ \zeta &\equiv \Delta E_{31} U_{\mu 3} |U_{e3}|, \end{aligned}$$

we can rewrite $Y_2^{\mu e} = \zeta e^{i\delta}$ and $Y_3^{\mu e} = \xi e^{i\delta} + \eta e^{-2i\gamma}$, where $\delta$ is the Dirac CP phase of the MNS matrix $U$, so we have

$$\begin{aligned} \tilde{X}_1^{\mu e} &= \frac{-e^{i\delta}}{\Lambda_-(\Lambda_+ - \Lambda_-)} [\xi + \eta e^{-i(2\gamma+\delta)} - \Lambda_+ \zeta] \\ \tilde{X}_2^{\mu e} &= \frac{e^{i\delta}}{\Lambda_+\Lambda_-} [\xi + \eta e^{-i(2\gamma+\delta)} - (\Lambda_+ + \Lambda_-) \zeta] \\ \tilde{X}_3^{\mu e} &= \frac{e^{i\delta}}{\Lambda_+(\Lambda_+ - \Lambda_-)} [\xi + \eta e^{-i(2\gamma+\delta)} - \Lambda_- \zeta]. \end{aligned}$$

Notice that the phase factor $e^{i\delta}$ in front of each $\tilde{X}_j^{\mu e}$ drops out in the oscillation probability $P(\nu_\mu \to \nu_e)$ because $P(\nu_\mu \to \nu_e)$ is expressed in terms of $\tilde{X}_j^{\mu e} \tilde{X}_k^{\mu e *}$, and the oscillation probability (23) depends only on the combination $2\gamma + \delta = \arg(\epsilon_{e\mu}) + \delta$.

In the present case, the matrix $\tilde{U}$ is unitary and because of this three flavor unitarity all the T violating terms are proportional to one factor:

$$\begin{aligned} & 2 \sum_{j<k} \mathrm{Im} \left( \tilde{X}_j^{\mu e} \tilde{X}_k^{\mu e *} \right) \sin \left( \Delta \tilde{E}_{jk} L \right) \\ =\ & 2\, \mathrm{Im} \left( \tilde{X}_1^{\mu e} \tilde{X}_2^{\mu e *} \right) [\sin \left( \Delta \tilde{E}_{12} L \right) - \sin \left( \Delta \tilde{E}_{13} L \right) + \sin \left( \Delta \tilde{E}_{23} L \right)] \\ =\ & -8\, \mathrm{Im} \left( \tilde{X}_1^{\mu e} \tilde{X}_2^{\mu e *} \right) \sin \left( \frac{\Delta \tilde{E}_{21} L}{2} \right) \sin \left( \frac{\Delta \tilde{E}_{31} L}{2} \right) \sin \left( \frac{\Delta \tilde{E}_{32} L}{2} \right). \end{aligned}$$

This modified Jarlskog factor $\mathrm{Im}(\tilde{X}_1^{\mu e}\tilde{X}_2^{\mu e*})$ in matter can be rewritten as

$$\begin{aligned}\mathrm{Im}(\tilde{X}_1^{\mu e}\tilde{X}_2^{\mu e*}) &= \frac{1}{\Lambda_+\Lambda_-(\Lambda_+-\Lambda_-)}\mathrm{Im}(Y_3^{\mu e}Y_2^{\mu e*}) = -\frac{\eta\zeta\sin(2\gamma+\delta)}{\Lambda_+\Lambda_-(\Lambda_+-\Lambda_-)}\\ &= -\frac{A(\Delta E_{31})^2}{\Lambda_+\Lambda_-(\Lambda_+-\Lambda_-)}\,|\epsilon_{e\tau}X_3^{e\mu}X_3^{\mu\tau}|\,\sin(\arg(\epsilon_{e\mu})+\delta).\end{aligned}$$

This completes derivation of Eq. (23).

## ACKNOWLEDGMENTS

The author would like to thank Alexei Smirnov for bringing my attention to Refs. [5, 6, 7, 8]. He would also like to thank He Zhang for calling my attention to Refs. [4, 10] which were missed in the first version of this paper. This research was supported in part by a Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, #19340062.

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